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LETTER TO THE EDITOR

Phase separation in the spherical model

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Abstract. The structure of the interface between phases in the spherical model of a ferromagnet is solved exactly.

The structure of the interface between phases has recently been subjected to considerable scrutiny (Fisk and Widom 1969, Widom 1972). A central feature is the realisation that the interface profile might well become diffuse, unless stabilised by gravity, for instance; this occurs because of thermal excitation of long-wavelength fluctuations. Such a phenomenon can be appreciated by studying the linearised drumhead model of an interface (Widom 1972), or capillary wave theories without cut-off (Buff *et al* 1965). Its existence has recently been confirmed within the framework of the renormalisation group (Jasnow and Rudnick 1978, Ohta and Kawasaki 1977) for systems with $n = 1$, where n is the number of local degrees of freedom involved.

In this Letter we present a rigorous analysis of phase separation in the spherical model of ferromagnetism introduced by Berlin and Kac (1952) and reviewed by Joyce (1972). Unlike the Ising model from which it is derived, this model can be solved exactly in all dimensions using a suitable translational symmetry breaking magnetic field h to achieve phase separation. As $h > 0$, the interface becomes diffuse; further, the surface tension tends to zero, recapturing the result of Barber and Fisher (1973).

A number of relevant rigorous results are known for d-dimensional lattice gases and their Ising-model equivalents. For $d = 2$, the profile without stabilisation has been obtained explicitly (Abraham and Reed 1974, 1976); the interface is rough, meaning that the magnetisation does not change on the scale of the lattice spacing, but it can be localised by suitable 'pinning' forces (Abraham 1979). When $d = 3$ van Beijeren (1975) has extended the result of Dobrushin (1972) to show that for $T < T_c$ (2) the interface has finite width. On the other hand, convincing evidence has been given (Weeks *et al* 1973) that the interface is rough for $T > T_R \sim T_c(2)$, as predicted by Burton *et al* (1951) on heuristic grounds. These authors suggested on the basis of a mean field argument that a roughening transition should occur at $T_c(d-1)$ in a d-dimensional system; *the spherical model does not have a roughening transition for any d.*

The spherical model has the same free energy per site as the classical rotor problem in the $n \to \infty$ limit (Stanley 1968, Kac and Thompson, unpublished). Here we have spins $\boldsymbol{S}_i$ at each site, each with n components and length $n^{1/2}$ coupled by a Heisenberg interaction. The case $n = 3$ is the classical Heisenberg ferromagnet. The analogy

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Pursuing this analogy, the interface between phases discussed here may be analogous to the Bloch wall of the associated ferromagnet; this has not been proven. However, there is a phenomenological theory which suggests that the Bloch wall should be unstable unless it is pinned (Kittel 1971; see pp 566–7 for a summary of phenomenology). Our results for the interface profile agree with this.

The model is defined as follows. Consider a d-dimensional lattice with unit sides centred on the origin, the vertices being labelled $\boldsymbol{r} = (\boldsymbol{x}, z)$ where $\boldsymbol{x}$ labels position in the $(d-1)$-dimensional sheet at height z. At the vertex $\boldsymbol{r}$ there is a spin $s(\boldsymbol{r})$. The energy of a spin configuration $\{s\}$ is

$$E(\{s\}) = -J \sum s(\boldsymbol{r})s(\boldsymbol{t}) - \sum h(\boldsymbol{r})s(\boldsymbol{r}) \tag{1}$$

where the first sum is over neighboring pairs. The lattice is taken to be cylindrical with axis $(\mathbf{0}, 1)$. Ferromagnetism is specified by positive J. The probability of $\{s\}$ is given canonically by

$$p(\{s\}) = Z^{-1} \exp(-\beta)E(\{s\}) \tag{2}$$

where $\beta = 1/kT$ is the usual notation. For the Ising model $s(\boldsymbol{r}) = \pm 1$. Berlin and Kac (1952) replaced this restriction by

$$-\infty < s(\boldsymbol{r}) < \infty \qquad \sum s(\boldsymbol{r})^2 = N \tag{3}$$

where N is the number of lattice sites. This permits one to exploit the gaussian character of (1) and (2). The field $h(\boldsymbol{r})$ is taken to be

$$h(\boldsymbol{r}) = h[(1-\delta(z))] \operatorname{sgn} z. \tag{4}$$

In the infinite lattice limit the top of the cylinder should be in the positively magnetised phase and vice versa. As $h \to 0+$ one should then select the spontaneous magnetisation which might, of course, vanish.

The detailed calculation of the following results, which involve nothing more than gaussian integration and careful implementation of (3), will be given elsewhere.

Magnetisation profile. We have

$$\begin{aligned} s(\mathbf{0}, z) &= (h/\pi\beta J) \int_0^{2\pi} d\omega \sin \omega z \cot(\tfrac{1}{2}\omega)[\zeta - (d-1) - \cos \omega]^{-1} \\ &= \operatorname{sgn} z \, m(h, \beta)\{1 - \exp(-v_0|z|)\} \end{aligned} \tag{5}$$

where

$$m(h, \beta) = h/2\beta J(\zeta - d) \tag{6}$$

is the magnetisation of a homogeneous phase in a field h. The saddle point ζ (Berlin and Kac 1952) is given by the solution of

$$2\beta J = [h/\beta J(\zeta - d)]^2 + 2f_d(\zeta) \tag{7}$$

with ζ real, $\zeta > d$. the function $f_d(\zeta)$ is given by

$$f_d(\zeta) = (2\pi)^{-d} \int_{-\pi}^{\pi} dq_1 \ldots \int_{-\pi}^{\pi} dq_d \left(\zeta - \sum_1^d \cos q_i\right)^{-1}. \tag{8}$$

Thus the saddle point has the same location as for an isotropic system with parameters (h, β). The function in (8) is monotone decreasing, and for $d \geqslant 3$ (integral!) remains finite as $\zeta \to d+$. Thus for $\beta > f_d(1)/J = \beta_c$ (the critical value) the saddle point behaves as $h \to 0$ as

$$\zeta - d \sim 2|h|[J(\beta - \beta_c)]^{-1/2}. \tag{9}$$

On the other hand, when $\beta < \beta_c$, $\zeta - d \not\to 0$ as $h \to 0$. From (6) there is a phase transition (Berlin and Kac 1952) to states with spontaneous magnetisation

$$\begin{aligned} m^* &= \lim_{h \to 0+} m(h, \beta) \\ &\sim [J(\beta - \beta_c)]^{1/2}/\beta_c 4J. \end{aligned} \tag{10}$$

There is no phase transition for $d = 1, 2$.

The length scale, v_0^{-1}, in (5) is given by

$$\sinh \tfrac{1}{2} v_0 = [\tfrac{1}{2}(\zeta - d)]^{1/2}. \tag{11}$$

Since $m^* > 0$ for $d \geqslant 3$ and $\beta > \beta_c$, and $v_0 \to 0$ as $h \to 0$ we conclude from (5) that the interface is rough; in particular, the result for $d \geqslant 4$ violates the criterion of Burton *et al* (1951) since there should be a roughening transition at $T_c(d-1) > 0$. Equations (7) and (9) show that in the critical region *the length scale is the correlation length of an isotropic phase with parameters* (h, β).

Our result does not agree precisely with the renormalisation group predictions, nor does it agree with the capillary wave model. Rather, it may well be an exactly solvable model related to a Bloch wall for $n \geqslant 2$.

Pair correlation function

$$\langle s(\mathbf{0}, z) s(\boldsymbol{x}, z)\rangle = \langle s(\mathbf{0}, x)\rangle^2 + f_2(\boldsymbol{x}, 0) \tag{12}$$

where

$$f_2(\boldsymbol{x}) = \int_{-\pi}^{\pi} d\omega_1 \ldots \int_{-\pi}^{\pi} d\omega_d \exp\left(i \sum_1^d x_i \omega_i\right)\left(\zeta - \sum_1^d \cos \omega_i\right)^{-1}$$

This result shows that a slab of matter at height z behaves as though it were taken from a homogeneous phase at a magnetisation $\langle s(\mathbf{0}, z)\rangle$ which lies between the extreme values. This type of decoupling is typical of an *ansatz* introduced in approximate theories (Brown and March 1976).

As $h \to 0$ in (12) the system assumes an infinite correlation length; in this model we cannot see an interplay between a length scale (v_0^{-1}) imposed by the field and an intrinsic correlation length because the latter diverges. On the other hand, there is some evidence that matter in the interface has anomalously long range correlations. (Wertheim 1976, Weeks 1977) along the interface; this is supported by (12).

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